Lecture 18 (2/18/22)

- Finish pf of Thm 1 from Lecture 17 notes.

Recall: We want to factor $f \in H(G)$ into $f = p \cdot g$, where $p, g \in H(G)$ and $g \neq 0$ and p being as "simple" as possible.

Def. (1) An elementary factor of order $p \in \{0, 1, 2, \dots\}$ is

$$\begin{cases} E_0(z) = 1 - z \\ E_p(z) = (1-z)e^{z + \frac{z^2}{2} + \dots + \frac{z^p}{p}}, & p \geq 1. \end{cases}$$

We first consider the case $G = \mathbb{C}$.

Construction Thm

Let $\{a_n\}_{n=1}^{\infty}$ be seq. in $\mathbb{C}$, $a_n \neq 0$ and $a_n \to \infty$, and $\{p_n\}_{n=1}^{\infty}$ a seq. of integers s.t. $\forall r > 0$

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n + 1} < \infty.$$

Then

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

converges in $H(\mathbb{C})$, and f's zero seq. is precisely $\{a_n\}_{n=1}^{\infty}$.

<u>Pf.</u> By previous thm, suffices to check that $\sum_{n=1}^{\infty} (1 - E_{p_n}(z/a_n))$ conv. abs + unif. (normally) on <u>every</u> $K \subset\subset \mathbb{C}$, and suffices to check $K = \overline{B(0,r)}$, $\forall r > 0$.

Thus, fix $r > 0$. We need to estimate $|1 - E_{p_n}(z/a_n)|$ for $|z| \le r$ and $n \gg 1$.

<u>Lemma</u>: For $|w| \le 1$, $p \ge 0$,

$$|1 - E_p(w)| \le |w|^{p+1}.$$

Suppose we can prove this lemma, then the conclusion follows since

$a_n \to \infty \Rightarrow \exists N$ s.t. $|w| = \left|\frac{z}{a_n}\right| \le 1$

when $|z| \le r$ and $n \ge N \Rightarrow n, m \ge N$

$$\sum_{j=n}^{m} |1 - E_{p_j}(\tfrac{z}{a_j})| \le \sum_{j=n}^{m} \left|\frac{z}{a_j}\right|^{p_j+1}$$

$$\le \sum_{j=1}^{m} \left(\frac{r}{|a_j|}\right)^{p_j+1} \to 0 \text{ as } n, m \to \infty \text{ by}$$

assumption.

Thus, suffices to prove lemma.

<u>Pf of Lemma</u>. Clearly, conclusion holds for $p=0$. Suppose $p \ge 1$;

$E_p(w) = (1-w)\, e^{w + \dots + w^p/p}$ and

$E_p'(w) = -w^p e^{w + \dots + w^p/p}$.

Consider $g(w) = E_p(w) - 1$ in $|w| < 1$

Obs. $g(0) = 0$ and $g'(0) = \dots = g^{(p)}(0) = 0$.

(zero of order $\ge p+1$)

Now, the Taylor series of $e^{w+\dots+w^p/p}$ has only ≥ 0 coefficients. $\Rightarrow$

Taylor series of $E_p'(w)$ has all ≤ 0 coeff's. Thus, we have

$$E_p(w) = 1 + \sum_{j=p+1}^{\infty} e_j w^j$$

where $e_j \leq 0$ $\Rightarrow \left| E_p(w) - 1 \right| \leq |w|^{p+1} \sum_{j=p+1}^{\infty} |e_j|$

Moreover $\sum_{j=p+1}^{\infty} |e_j| = -\sum_{j=p+1}^{\infty} e_j \cdot 1^j =$

$-(E_p(1) - 1) = 1 \Rightarrow$

$\left| E_p(w) - 1 \right| \leq |w|^{p+1}$ as desired. ∎

Prop 1. For $\{a_n\}_{n=1}^{\infty}$ as in Thm 1,

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^n < \infty .$$

Rem. Thus, $p_n = n-1$ in Thm 1 always works.

Pf. $a_n \to \infty \Rightarrow \frac{r}{|a_n|} \le \frac{1}{2}$ for $n \gg 1$. $\blacksquare$

We then have.

Weierstrass Factorization Thm. Let f be entire function, $\{a_n\}_{n=1}^{\infty}$ its seq. of zeros (w/ multipl.) that do not include any zero at $z=0$. Let m be the order of a (possible) zero at $z=0$ (or $m=0$ if no zero). Then $\exists$ $\{p_n\}_{n=1}^{\infty}$ seq. of ≥ 0 integers, g entire function s.t.

$$f(z) = z^m \prod_{n=1}^{\infty} E_{p_n}(z/a_n)\, e^{g(z)} .$$

Pf. DIY. $\blacksquare$

Ex. (1) Factor $g(z) = \sin \pi z$.

- **Zeros.** $f(z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z}) \Rightarrow$

$f(z) = 0 \Leftrightarrow e^{2\pi i z} = 1 \Rightarrow z = k, \; k \in \mathbb{Z}$.

We consider $\tilde{g}(z) = \frac{\sin \pi z}{z}$, also entire w/ zeros at $k \in \mathbb{Z} \setminus \{0\}$.

- Since $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$, $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$,

we may take $p_n = 1$ in Constr. Thm.

$f(z) = \prod_{k \in \mathbb{Z} \setminus \{0\}} E_1(z/k) =$

$\lim_{n \to \infty} \prod_{k=1}^{n} (1 - \frac{z}{k}) \cancel{e^{z/k}} (1 + \frac{z}{k}) \cancel{e^{-z/k}}$

$= \lim_{n \to \infty} \prod_{k=1}^{n} \left(1 - \frac{z^2}{k^2}\right) = f(z)$

$$\Rightarrow \sin \pi z = z \prod_{k=1}^{\infty} \left(1 - z^2/k^2\right)$$